Numerical modeling of the linear relaxational filtration by Monte Carlo methods

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Abstract. Four models of linear relaxational filtration is considered. Initial and boundary conditions are set for them (Dirichlet, Neumann and mixed). Obtained problem solved by Monte Carlo methods – "random walk on spheres", "random walk on balls" and "random walk on lattices" of Monte Carlo methods and by probability difference methods.

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1. Introduction

The linear relaxational filtration is described by the conservation law of pulse of resistance force, by the linearized conservation law of a fluid mass and determining relations for pulse of resistance forces and fluid mass. After exception of a pulse density of resistance forces ($\mathbf{J}$) and ($m\rho$) this system with respect to pressure (p) and velocity of filtration ($\mathbf{W}$) is

$$\Delta p(x,t) = \frac{F(0)\Phi(0)}{\rho_0}\frac{\partial^2 p(x,t)}{\partial t^2} + \int\limits_0^\infty \left(\frac{F(0)}{\rho_0}\frac{d\Phi(t')}{dt'} + \frac{\Phi(0)}{\rho_0}\frac{dF(t')}{dt'} + \right.$$
$$\left. \frac{1}{\rho_0}\int\limits_0^{t'} \frac{dF(\tau)}{d\tau}\frac{d\Phi(t'-\tau)}{d(t'-\tau)}d\tau \right) \frac{\partial^2 p(x,t-t')}{\partial(t-t')^2}dt', \quad (1.1)$$

$$-F(0)\frac{\partial \mathbf{W}(x,t)}{\partial t} - \int\limits_0^\infty \frac{dF(t')}{dt'}\frac{\partial \mathbf{W}(x,t-t')}{\partial(t-t')}dt' = \operatorname{grad}_x p(x,t). \quad (1.2)$$

Here $F(t)$ and $\Phi(t)$ are relaxation kernels of the filtration law and fluid mass. [1].

We consider four models of relaxational filtration.

I. A model of classical elastic filtration. This model of filtration relates to the kernels of relaxation $F(t) = \frac{\mu}{\kappa} t\eta(t)$, $\Phi(t) = \rho_0 \beta \eta(t)$, and equations (1.1) and (1.2) take form

$$\chi \Delta p(x,t) = \frac{\partial p(x,t)}{\partial t}, \tag{1.3}$$

$$\mathbf{W}(x,t) = -\frac{\kappa}{\mu} \operatorname{grad}_x p(x,t). \tag{1.4}$$

II. The simplest model of filtration with a constant speed of disturbance spread. This model is defined with kernels of relaxation: $F(t) = \frac{\mu}{\kappa}(t+\tau)\eta(t)$, $\Phi(t) = \rho_0 \beta \eta(t)$. For the given model the system (1.1) – (1.2) has a form:

$$\chi \Delta p(x,t) = \frac{\partial p(x,t)}{\partial t} + \tau \frac{\partial^2 p(x,t)}{\partial t^2}, \tag{1.5}$$

$$\tau \frac{\partial \mathbf{W}(x,t)}{\partial t} + \mathbf{W}(x,t) = -\frac{\kappa}{\mu} \operatorname{grad}_x p(x,t). \tag{1.6}$$

III. Filtration model in relaxationaly-compressed porous environment realized by the linear Darcy law. Corresponding kernels are $F(t) = \frac{\mu}{\kappa} t\eta(t)$, $\Phi(t) = \rho_0 \Big(\beta - \frac{\lambda_m - \lambda_p}{\lambda_m} \beta_c \exp\big(-\frac{t}{\lambda_m}\big)\Big)\eta(t)$. System (1.1) – (1.2) has a form:

$$\chi \Delta \left(p(x,t) + \lambda_m \frac{\partial p(x,t)}{\partial t} \right) = \frac{\partial}{\partial t} \left(p(x,t) + \lambda'_m \frac{\partial p(x,t)}{\partial t} \right), \tag{1.7}$$

$$\mathbf{W}(x,t) = -\frac{\kappa}{\mu} \operatorname{grad}_x p(x,t). \tag{1.8}$$

where $\lambda'_m = \lambda_m \frac{\beta_*}{\beta}$. In a particular case of incompressible fluid, $\beta_f = 0$ and $\lambda_p = 0$, instead of (1.7) – (1.8) we have

$$\chi \Delta \left(p(x,t) + \lambda_m \frac{\partial p(x,t)}{\partial t} \right) = \frac{\partial p(x,t)}{\partial t}, \tag{1.9}$$

$$\mathbf{W}(x,t) = -\frac{\kappa}{\mu} \operatorname{grad}_x p(x,t). \tag{1.10}$$

Model (1.9) – (1.10) describes a filtration of incompressible fluid in relaxationaly-compressed porous environment for $\lambda_p = 0$, and also in fractured-porous environment with infinitesimal elasticity of fractures and conductivity of blocks.

IV. Model of filtration by the simplest unbalanced law in elastic porous environment. Here the kernels of relaxation have form: $F(t) = \frac{\mu}{\kappa} \Big(t - \big(t_W - t_p\big)\Big(1 - \exp\big(-\frac{t}{\tau_p}\big)\Big)\Big)\eta(t)$, $\Phi(t) = \rho_0 \beta \eta(t)$. For this model system (1.1) – (1.2) lead to form:

$$\chi \Delta \left(p(x,t) + \tau_p \frac{\partial p(x,t)}{\partial t} \right) = \frac{\partial}{\partial t} \left(p(x,t) + \tau_W \frac{\partial p(x,t)}{\partial t} \right), \tag{1.11}$$

$$\tau_W \frac{\partial \mathbf{W}(x,t)}{\partial t} + \mathbf{W}(x,t) = -\frac{\kappa}{\mu} \operatorname{grad}_x \left(p(x,t) + \tau_W \frac{\partial p(x,t)}{\partial t} \right). \tag{1.12}$$

We describe functions and parameters incoming in four filtration models. τ is time of relaxation, κ is penetrability coefficients, m is a porosity, t is time, β is elasticity capacity coefficient of the layer, $\beta = \beta_c + m_0\beta_f$, β_c is compressibility coefficient of the porous environment, m_0 is a fluid porosity in the unperturbed layer conditions, β_f is compressibility coefficient of the fluid, μ is a fluid viscosity, ρ is a fluid density, $\chi = \frac{\kappa}{\mu\beta}$ is piezoconductivity coefficient of the layer, $\eta(t)$ is Heaviside function, $\eta(t) = 1$ for $t > 0$, $\eta(t) = 1/2$ for $t = 0$, $\eta(t) = 0$ for $t < 0$, τ_W and τ_p nonnegative constants relaxation times of filtration velocity and pressure, $\lambda'_m = \lambda\frac{\beta_*}{\beta}$, $\beta_* = m_0\beta_f + \beta_c\lambda_p/\lambda_m$ is dynamics coefficient of elasticity capacity, λ_m is the relaxation time of porosity under the constant overfull of pressure, λ_p is the relaxation time of pressure under the constant porosity, p_0 is pressure in the unperturbed layer conditions, ρ_0 is density in the unperturbed layer conditions. All parameters are nonnegative given numbers.[1].

Mathematical problems for models I – IV. Initial conditions for all four models. First of all in all four models in bounded region of filtration $\Omega \in \mathbb{R}^3$ with boundary $\partial\Omega$ and for $t \in [0,T]$ and for pressure $p(x,t)$ we consider equations (1.3), (1.5), (1.7), (1.9) and (1.11). Then we set initial conditions for them. For equations (1.3) and (1.9):

$$p(x,t) = a(x), \text{while } \ t = 0, \tag{1.13}$$

and for equations (1.5), (1.7) and (1.11) besides condition (1.13) we give an additional condition

$$\frac{\partial p(x,t)}{\partial t} = b(x), \text{while } \ t = 0. \tag{1.14}$$

Boundary conditions for all four models.

Problem 1. (Dirichlet Problem). *In bounded filtration region $\Omega \in \mathbb{R}^3$ with boundary $\partial\Omega$ and for time $t \in [0,T]$, function $p(x,t)$ satisfies the boundary condition*

$$p(x,t) = p_1(x,t) \ \text{ for } \ x \in \partial\Omega \times [0,T]. \tag{1.15}$$

Problem 2. (Neumann Problem). *In bounded filtration region $\Omega \in \mathbb{R}^3$ with boundary $\partial\Omega$ and for time $t \in [0,T]$, function $p(x,t)$ satisfies the boundary condition*

$$\frac{\partial p(x,t)}{\partial \mathbf{n}} = p_2(x,t) \ \text{ for } \ x \in \partial\Omega \times [0,T], \tag{1.16}$$

where $\mathbf{n}$ is an internal normal.

Problem 3. (Mixed Problem). *In bounded filtration region $\Omega \in \mathbb{R}^3$ with boundary $\partial\Omega$ and for time $t \in [0,T]$, function $p(x,t)$ satisfies the boundary condition*

$$\alpha_1 p(x,t) + \beta_1 \frac{\partial p(x,t)}{\partial \mathbf{n}} = p_3(x,t) \ \ for \ \ x \in \partial\Omega \times [0,T], \tag{1.17}$$

where $\mathbf{n}$ *is an internal normal.*

An idea for solving by Monte Carlo methods.

Initial-boundary problem with respect to pressure $p(x,t)$ is discretizied only by variable t. For that, interval $[0,T]$ split onto N equal steps of $\Delta\tau = \frac{T}{N}$, $t_n = n\Delta\tau$, $n = 1,2,\ldots,N$. As a result we get discrete boundary problem by time variable. Obtained, the problems for elliptic type PDEs (Helmholtz equation) solved by Monte Carlo methods.

2. Solution of the initial boundary value problems by Monte Carlo methods

We demonstrate a solution of the initial boundary value problems by Monte Carlo methods on the following model – Filtration in relaxationaly-compressed porous environment realized by the linear Darcy law, that is model III. For this model we have a mathematical problem:

$$\chi\Delta\left(p(x,t) + \lambda_m \frac{\partial p(x,t)}{\partial t}\right) = \frac{\partial}{\partial t}\left(p(x,t) + \lambda'_m \frac{\partial p(x,t)}{\partial t}\right), \tag{2.1}$$

$$p(x,t) = a(x), \text{while} \ \ t = 0, \tag{2.2}$$

$$\frac{\partial p(x,t)}{\partial t} = b(x), \text{while} \ \ t = 0, \tag{2.3}$$

$$p(x,t) = p_1(x,t) \ \ \text{for} \ \ x \in \partial\Omega \times [0,T]. \tag{2.4}$$

2.1. Solution of the Dirichlet Problem (2.1) – (2.4)

Let coefficients χ, λ_m, λ'_m are while positive fixed values. Let us divide interval $t \in [0,T]$ into N equal parts with length $\Delta\tau$. So that $t_n = n \cdot \Delta\tau$, $n = 0,1,\ldots,N$, $\Delta\tau = \frac{T}{N}$, $\Delta\tau > 0$, and we digitize only with respect to t using implicit scheme. In result taking into account λ'_m, we obtain the equation (2.1) on time layer t_{n+1}

$$\Delta p^{n+1}(x) - a_1 \cdot p^{n+1}(x) = f^n(x), \tag{2.5}$$

where $f^n(x) = b_1 \cdot p^n(x) + c_1 \cdot \Delta p^{n-1}(x) + d_1 \cdot p^{n-1}(x)$, $c_1 = \dfrac{\lambda_m}{2\Delta\tau + \lambda_m}$,

$a_1 = \dfrac{m_0\beta_f(\Delta\tau + 2\lambda_m) + \beta_c(\Delta\tau + 2\lambda_p)}{\wp}$, $b_1 = -\dfrac{4(m_0\beta_f\lambda_m + \beta_c\lambda_p)}{\wp}$,

$d_1 = \dfrac{m_0\beta_f(2\lambda_m - \Delta\tau) + \beta_c(2\lambda_p - \Delta\tau)}{\wp}$, $\quad \wp = \Delta\tau\chi(2\Delta\tau + \lambda_m) \cdot (\beta_c + m\beta_f)$.

The algorithm "Random walk on spheres" of Monte Carlo methods. It is clear that $a_1 > 0$, as parameters $m_0,\ \beta_f,\ \Delta\tau,\ \lambda_m,\ \beta_c,\ \lambda_p,\ \chi$ are positive. Combining the initial condition with (2.5) we obtain

$$p^0(x) = a(x), \quad x \in \Omega, \quad \frac{p^1(x) - p^0(x)}{\Delta\tau} = b(x), \quad x \in \Omega, \tag{2.6}$$

which are the difference analogues of the initial data (2.2) and (2.3) respectively. For this problem the boundary condition transformed to:

$$p^{n+1}(x) = p_1^{n+1}(x), \quad x \in \partial\Omega. \tag{2.7}$$

We shall call the boundary $\partial\Omega$ (and $\partial\Omega_\varepsilon$) satisfying the Dirichlet condition as absorbing boundary. It is known that the problem (2.5) – (2.7) (Dirichlet problem for the Helmholtz equation of a time layer t_{n+1}), is solved with the help of "random walk on spheres" algorithm of Monte Carlo methods. The constructed ε–displaced estimation of the solution $p^{n+1}(x)$ with the help of "random walk on spheres" algorithm has a uniformly bounded variance by ε. [2], [3], [4], [5], [6], [7].

The algorithm "Random walk on balls" of Monte Carlo methods. "Random walk on balls" algorithm for solving a Dirichlet problem. This algorithm is similar to algorithm "random walk on spheres". In algorithm "random walk on balls" a "particle" passes from the center of the ball to a random point inside the ball and including a bound of ball (sphere), that is the following state of Markov chain inside the ball and including a bound of ball. It can be proved that Markov's chain converges in the same manner as for "random walk on spheres" algorithm and for finite number of steps to the ε-bound of $\partial\Omega_\varepsilon$. But it is obvious that convergence of Markov chain for "random walk on balls" algorithm is slower than for "random walk on spheres". For that reason, "random walk on balls" algorithm is almost unusable for numerical modeling by Monte Carlo methods. The constructed ε-displaced estimation of the solution $p^{n+1}(x)$ with the help of "random walk on balls" algorithm has a uniformly bounded variance by ε. [5].

The algorithm "Random walk on lattices" of Monte Carlo methods. At first we approximate the solution (2.5) – (2.7) with the help of finite difference method and construct Markov chain, its transition probabilities are defined with the help of coefficients and parameters of the difference problem (2.5) – (2.7). For this purpose we use the following approximation of the second derivative with respect to x, i.e. $p^{n+1}_{x_i x_i}(x) = \dfrac{p^{n+1}(x + e_i\,h) + p^{n+1}(x - e_i\,h) - 2p^{n+1}(x)}{h^2}$, where h is step along x, e_i is the unit vector along the axis x_i. Obviously $\mathrm{O}(h^2)$ is a precision of the such approximation. Let's denote approximation of a domain Ω by ω_h, and boundary $\partial\Omega$ – by γ_h. Now by time lowering superscripts $n+1$, n, $n-1$ from (2.5), we obtain the following finite difference equation

$$p(x_i) = \frac{1}{2 + a_1\,h^2} \cdot p(x_i + e_i\,h) + \frac{1}{2 + a_1\,h^2} \cdot p(x_i - e_i\,h) - \frac{h^2}{2 + a_1\,h^2} \cdot f(x_i). \tag{2.8}$$

It's obvious that

$$\frac{2}{2+a_1\,h^2} \longrightarrow 1 \;\text{ for }\; h\to 0,\;\; \tau\to,\;\; \lambda_m\to 0, \tag{2.9}$$

where h is step along x, τ is time step. That is realization of (2.9) correspond to convergence requirements of a difference schemes and relaxation process. Let's denote $\alpha(x_i,y_i,h,\Delta\tau)=\dfrac{1}{2+a_1\,h^2}$. As $\alpha(x_i,y_i,h,\Delta\tau)>0$ and $\alpha+\alpha\le 1$ on y_i for $\forall x_i$, then $\alpha(x_i,y_i,h,\Delta\tau)$ are transition probabilities of Markov chain. Here $y_i=x_i\pm e_i\,h$, e_i is unit vector.

Algorithm. At first we play a coordinate axis with probability $1/3$ for $\Omega\in\mathbb{R}^3$. Then the "particle" moves (along the direction $-e_i$ or $+e_i$) with identical probability α from node x_i into one of the neighboring node $x_i\pm e_i\,h$. It is necessary to take into account the "weight" of node, it is proportional to $\dfrac{h^2}{2+a_1\,h^2}\cdot f(x_i)$. And so on until the "particle" achieves the discrete boundary γ_h. As soon as the "particle" achieves the boundary γ_h, boundary data $p_1(x_i)$ is add to a counter. Thus a random variable $\xi^h_{N_h}$ is defined along a discrete Markov chain with random length N_h. Then we average it on all trajectories, that is the estimation of the solution $p^{n+1}(x_i)$ in the node x_i is defined from $p^{n+1}(x_i)\approx\dfrac{1}{M}\sum\limits_{i=1}^{M}\left(\xi^h_{N_h}\right)_i$, where M is trajectories amount of Markov chain starting from the node x_i. [8], [10], [7], [9].

Then we have the following

Theorem 1. *The Neumann–Ulam scheme is applicable to the finite difference problem for (2.5) – (2.7).*

Proof. Proof of the theorem follows from algorithm of the discrete solution of promlem (2.5) – (2.7). The complete proof see in [10]. The theorem is proved. □

In this case variance of an estimation of the solution $p^{n+1}(x_i)$ will be bounded, it can be explicitly calculated. [10], [8], [5].

Probability difference method. Let's consider the finite difference problem (2.8) for a time layer $n+1$ with a discrete boundary condition $p(x_i)=p_1(x_i)\quad x_i\in\gamma_h$. Let's denote by $\left\{\zeta^h_i,\;i=0,1,\dots\right\}$ value of transition chain. Let $p_1(x)$ is the arbitrary continuous function for $x\in\gamma_h$. Let N_h is a moment of the first way out of a discrete domain ω_h: $N_h=\min\left\{i:\;\zeta^h_i\notin\omega_h\right\}$. Combining (2.8) with a boundary condition we obtain

$$p(x)=\mathbf{E}_x p(\zeta^h_1)+\triangle t^h\,\alpha\,f(x),\quad x\in\omega_h,\quad p(x)=p_1(x),\quad x\in\gamma_h \tag{2.10}$$

If $\mathbf{E}_x\,N_h<\infty$, then the problem (2.10) has a unique solution

$$p_h(x)=\mathbf{E}_x\Big\{\sum_{i=0}^{N_h-1} f(\zeta^h_i)\cdot\triangle t^h_i+p_1(\zeta^h_{N_h})\Big\}. \tag{2.11}$$

Here $\triangle t_i^h = \triangle t^h(\zeta_i^h)$ is a process parameter. If $f(x) = 0$, $\mathbf{P}_x\Big\{N_h < \infty\Big\} = 1$, then (2.10) has the unique solution

$$p_h(x) = \mathbf{E}_x\Big\{p_1(\zeta_{N_h}^h) \cdot I_{\{N_h<\infty\}}\Big\}. \tag{2.12}$$

[11], [12], [13], [14], [7].

2.2. Solution of the Neumann Problem 2

Let's consider (2.5) with initial conditions (2.6) and boundary conditions

$$\frac{\partial p^{n+1}(x)}{\partial \mathbf{n}} = p_2^{n+1}(x), \qquad x \in \partial\Omega. \tag{2.13}$$

The boundary $\partial\Omega$ (and $\partial\Omega_\varepsilon$), that correspond to the Neumann condition is called the reflecting boundary.

The algorithm "Random walk on spheres" of Monte Carlo methods. Let the solution of the problem (2.5) – (2.6), (2.13) is defined in a point $x_0 \in \Omega_\varepsilon$, where $\Omega_\varepsilon \subset \Omega$ is a domain with the boundary $\partial\Omega_\varepsilon$. $\partial\Omega_\varepsilon$ is ε–vicinity of the boundary Ω. State of Markov chain $\{x_i\}$ is defined with the help of the "random walk by spheres" process, by reaching $\partial\Omega_\varepsilon$–boundary the "particle" is reflected from $\partial\Omega_\varepsilon$–boundary into previous point (chain returns to the state before reflection). The "particle" continues random walk. After reflection the "weight" of boundary proportional to $p_2^{n+1}(x)$ is add to the counter. The chain breaks with the given probability $\zeta(\varepsilon)$, it is "small", $\zeta(\varepsilon) \to 0 \quad$ for $\quad \varepsilon \to 0$. Here we shall note, that the "particle" moves to the $\partial\Omega_\varepsilon$–boundary along the normal $\mathbf{n}$ in "random walk on spheres" algorithm. We obtain ε–displaced estimation of a solution $p^{n+1}(x)$ of the problem (2.8), (2.6), (2.13) in point x by averaging of random variable η_{N_α} constructed along Markov chain of random length N_α. Probability error follows from the central limit theorem. It can be estimated as $\mathbf{P}\{\text{choice error } < \varepsilon\} \cong \operatorname{erf}\Big(\frac{|\epsilon|\sqrt{M/2}}{\sigma^2}\Big)$, where $\mathbf{P}$ denote probability the error is no more than $|\varepsilon|$, M is quantity of trajectories, σ^2 is sampling of variance. [15], [16], [7], [5], [17], [18].

The algorithm "Random walk on balls" of Monte Carlo methods. "Random walk on balls" algorithm for solving a Neumann problem. This algorithm works in the same way as " random walk on balls" algorithm for Dirichlet problem. In this case, when "particle" reaches ε-bound of $\partial\Omega_\varepsilon$, is reflected in previous point and modeling of Markov chain is continued. The chain breaks with the given probability $\zeta(\varepsilon)$, it is "small", $\zeta(\varepsilon) \to 0$ for $\varepsilon \to 0$. Here we shall note, that the "particle" moves to the $\partial\Omega_\varepsilon$-boundary along the normal $\mathbf{n}$ in "random walk on balls" algorithm. We obtain ε-displaced estimation of a solution $p^{n+1}(x)$ of the problem (2.8), (2.6), (2.13) in point x by averaging of random variable η_{N_α} constructed along Markov chain of random length N_α. Probability error follows from the central limit theorem. It can be estimated as $\mathbf{P}\{\text{choice error } < \varepsilon\} \cong \operatorname{erf}\Big(\frac{|\epsilon|\sqrt{M/2}}{\sigma^2}\Big)$, where $\mathbf{P}$ denote probability the error is no more than $|\varepsilon|$, M is quantity of trajectories, σ^2 is sampling of variance. [15], [16], [7], [5], [17], [18].

The algorithm "Random walk on lattices" of Monte Carlo methods. Just as in a case of Dirichlet problem we get finite difference Neumann problem for three-point difference equation on a time layer $n+1$, that is (2.8), (2.6), (2.13). Here condition (2.9) for α is realized, i.e. $\alpha(x_i, y_i, h, \tau)$ are transition probabilities of Markov chain.

Algorithm. At first we play coordinate axis with probability 1/3. Then the "particle" moves (along the direction $-e_i$ or $+e_i$) with equal probability α from the node x_i into one of a neighboring node $x_i \pm e_i\, h$. It's necessary to take into account the "weight" of node proportional $\dfrac{h^2}{2 + a_1\, h^2} \cdot f(x_i)$. And so on until the "particle" achieves a discrete boundary γ_h. By reaching γ_h–boundary the "particle" is reflected into previous point, and boundary data proportional $p_2(x_i)$ is add to the counter. Near the boundary $\partial\Omega$ a step of grid h^* along the direction to boundary γ_h such that the "particle" gets on discrete ε–boundary γ_h^ε. Random walk process continues. The chain breaks with the given probability $\zeta(\varepsilon)$, it is "small" value, $\zeta(\varepsilon) \to 0 \quad \text{for} \quad \varepsilon \to 0$. Thus, random variable $\eta^h_{N_h}$ is defined along a discrete Markov chain with random length N_h. Then we average it on all trajectories, that is the estimation of the solution $p^{n+1}(x_i)$ in node x_i is defined from $p^{n+1}(x_i) \approx \dfrac{1}{M} \sum\limits_{i=1}^{M} \big(\eta^h_{N_h}\big)_i$, where M is trajectories amount of Markov chain starting from the node x_i. [8], [10], [12], [13], [7], [11].

Probability difference method. Let's consider the problem (2.8), (2.6), (2.9). Let $p_2(x)$ is a real bounded continuous function on a set $\partial\Omega$. $\partial\Omega$ is reflecting boundary. Approximation of (2.13) gives

$$\big(d(\partial\Omega) \cdot \nabla\big) p(x) = p_2(x). \tag{2.14}$$

Let the set $\partial\Omega_h$ approximate $\partial\Omega$ "from within". That is either $x \in \overline{\Omega} \bigcap \mathbb{R}^3_h$ or $x \in \partial\Omega$ or straight line connecting x with one of the nearest node $x_i \pm e_i h$, $x_i \pm e_i h \pm e_j h$ or $x_i \pm e_i h \mp e_j h$ touches $\partial\Omega$. Then α gives transition probabilities of the approximating chain ξ^h_i in Ω_h. The chain breaks with the given probability $\zeta(\varepsilon)$, it is "small", $\zeta(\varepsilon) \to 0 \quad \text{for} \quad \varepsilon \to 0$. It should be noted that $\mathbf{E}_x \Big\{ \xi^h_{n+1} - \xi^h_n |\, \xi^h_n = y_i \in \partial\Omega_h \Big\} = \upsilon(y)\, h / |\upsilon(y)|$. It is coordinated that reflection from the point $\partial\Omega_h$ happens along the direction $\upsilon(y)$. $\upsilon(y)$ is direction of hit in interior node. $\upsilon(x) = \sum\limits_{i=1}^{3} |\upsilon_i(x)|$. Transition probabilities on $\partial\Omega^{\mathbb{R}}_h$: $\varrho_h(x, x \pm e_i\, h) = \upsilon_i^{\pm} / |\upsilon(x)|$. Let's define $A^h_n = \prod\limits_{i=0}^{n} \exp\big(- a(\xi^h_i) \cdot \triangle t^h_i \cdot I_{\Omega_h}(\xi^h_i)\big)$, where t^h_i is a discrete time, parameter of ξ^h_i process. For the chain with random length N_h we get unique discrete approximation of solution of the problem (2.8), (2.6), (2.9)

$$p_h(x) = \mathbf{E}_x \Big\{ \sum_{i=0}^{N_h - 1} A^h_i \cdot f(\xi^h_i) \cdot \triangle t^h_i \cdot I_{\Omega_h}(\xi^h_h) + \sum_{i=0}^{N_h - 1} A^h_i \cdot p_2(\xi^h_i) \cdot d\phi^h_i \Big\} \tag{2.15}$$

[11], [8], [13], [7].

2.3. Solution of the Mixed Problem 3

Let's consider the problem (2.5), (2.6). To this problem we'll connect approximation of the mixed boundary condition (1.17) on a time layer $n+1$

$$\alpha_1\, p^{n+1}(x) + \beta_2\, p^{n+1}(x) = p_3^{n+1}(x), \quad x \in \partial\Omega_\varepsilon, \tag{2.16}$$

where $\beta_2 = \beta_1\big(d(\partial\Omega)\cdot\nabla\big)$.

The algorithm "Random walk on spheres" of Monte Carlo methods. As in a case of the Dirichlet problem we construct Markov chain by "random walk on spheres". In general, by reaching $\partial\Omega_\varepsilon$–boundary of a domain Ω the "particle" is absorbed or reflected with equal probability 1/2. But in our case, if the "particle" is absorbed, then in each point the value of "weight" $p_3(x_i)/\alpha_1$ is added to Markov chain, and if the "particle"reflected then we add $p_3(x_i)/\beta_2$. The chain breaks if a "particle" is absorbed. We get ε–displaced estimation of the solution $p_\varepsilon(x)$ of the problem (2.8), (2.6), (2.16) in the point x by averaging random variable ξ_i constructed along Markov chain with random length N.That is $p_\varepsilon(x) = \dfrac{1}{M}\sum\limits_{i=1}^{M}\xi_i$. [19], [20], [13], [7], [12].

The algorithm "Random walk on balls" of Monte Carlo methods. "Random walk on balls" for solving a mixed problem. As in a case of the Dirichlet problem we construct Markov chain by "random walk on balls". In general, by reaching $\partial\Omega_\varepsilon$-boundary of a domain Ω the "particle" is absorbed or reflected with equal probability 1/2. But in our case, if the "particle" is absorbed, then in each point the value of "weight" $p_3(x_i)/\alpha_1$ is added to Markov chain, and if the "particle"reflected then we add $p_3(x_i)/\beta_2$. The chain breaks if a "particle" is absorbed. We get ε-displaced estimation of the solution $p_\varepsilon(x)$ of the problem (2.8), (2.6), (2.16) in the point x by averaging random variable ξ_i constructed along Markov chain with random length N. That is $p_\varepsilon(x) = \dfrac{1}{M}\sum\limits_{i=1}^{M}\xi_i$. [19], [20], [13], [7], [12].

The algorithm "Random walk on lattices" of Monte Carlo methods. Let's consider the following finite difference problem (2.8), (2.6)

$$\alpha_1\, p(x_i) + \beta_2\, p(x_i) = p_3(x_i), \quad x_i \in \gamma_h. \tag{2.17}$$

The problem (2.8), (2.6), (2.17) is considered on a time layer $n+1$.

Algorithm. At first we play coordinate axis with probability 1/3 for $\Omega \in \mathbb{R}^3$. Then the "particle" moves (along the direction $-e_i$ or $+e_i$) with equal probability α from the node x_i into one of a neighboring node $x_i \pm e_i\, h$. It is necessary to take into account the "weight" of node, it proportional $\dfrac{h^2}{2 + a_1\, h^2}\cdot f(x_i)$. And so on until the "particle" achieves the discrete boundary γ_h. In general, on a boundary γ_h the "particle" is absorbed or reflected with equal probability 1/2. But in our case, the chain breaks if the "particle" is absorbed, and we add to counter a "weight" of absorbing boundary node $p_3(x_i)/\alpha_1$, at reflection – $p_3(x_i)/\beta_2$. Thus, we define a

random variable $\xi^h_{N_h}$ along a discrete Markov chain with random length N_h. The estimation of solution $p_h(x_i)$ in a node x_i is defined by $p_h(x_i) \approx \dfrac{1}{M}\sum\limits_{i=1}^{M}\left(\xi^h_{N_h}\right)_i$, where M is trajectories amount of Markov chain starting from the node x_i. [8], [13], [12], [10], [7], [19], [14].

Probability difference method. The problem (2.8), (2.6), (2.17) is considered on a time layer $n+1$. Let $p_3(x)$ is the real bounded continuous function on a set $\partial\Omega$. Let the set $\partial\Omega^{\mathbb{R}}_h$ approximate $\partial\Omega$ "from within". That is either $x \in \overline{\Omega}\bigcap\mathbb{R}^3_h$ or $x \in \partial\Omega$ or straight line connecting x with one of the nearest node $x_i \pm e_i h,\ x_i \pm e_i h \pm e_j h$ or $x_i \pm e_i h \mp e_j h$ touches $\partial\Omega$. The set is determined in $\overline{\Omega}\bigcap\mathbb{R}^3_h$. Let's define digitization $\Omega_h = \Omega\bigcap\mathbb{R}^3_h - \partial\Omega^{\mathbb{R}}_h$ of interior Ω and digitization of a stopping set $\partial\Omega^A_h = \mathbb{R}^3_h - \Omega_h - \partial\Omega^{\mathbb{R}}_h$. Then α gives transitive probabilities of the approximating chain ξ^h_i in Ω_h. The chain breaks at the first contact with $\partial\Omega^A_h$. Let's notice that $\mathbf{E}_x\left\{\xi^h_{n+1} - \xi^h_n |\, \xi^h_n = y_i \in \partial\Omega^{\mathbb{R}}_h\right\} = \upsilon(y)\, h/|\upsilon(y)|$. It is coordinated that the reflection from the point $\partial\Omega^{\mathbb{R}}_h$ happens along direction $\upsilon(y)$. $\upsilon(y)$ is the direction of hit into interior node. Let's define $A^h_n = \prod\limits_{i=0}^{n}\exp\big(-a(\xi^h_i)\cdot\triangle t^h_i\cdot I_{\Omega_h}(\xi^h_i)\big)$, $C^h_n = \prod\limits_{i=0}^{n}\exp\big(-\beta_1(\xi^h_i)\, d\phi^h_i\big)$, $D^h_n = A^h_n\, C^h_n$. We consider the case $\alpha_1 = \alpha_1(x)$, $\beta_1 = \beta_1(x)$, t^h_i is a discrete time, parameter of the process ξ^h_i, $d\phi^h = h\,/|\upsilon(x)|$, $d\phi^h_i = d\phi^h(\xi^h_i)\, I_{\partial\Omega^R_h}(\xi^h_i)$. For the chain with random length $N_h = \min\left\{n :\ \xi^h_n \in \partial\Omega^A_h\right\}$ we obtain unique discrete approximation of a solution of the problem (2.8), (2.6), (2.17)

$$p_h(x) = \mathbf{E}_x\Big\{\sum_{i=0}^{N_h-1} D^h_i\cdot f(\xi^h_i)\cdot\triangle t^h_i\cdot I_{\Omega_h}(\xi^h_i) + D^h_{N_h-1}\,\alpha_1(\xi^h_{N_h}) + \sum_{i=0}^{N_h-1} D^h_i\cdot p_3(\xi^h_i)\cdot d\phi^h_i\Big\}. \tag{2.18}$$

[11], [8], [13], [12], [10], [7], [19], [14].

3. Solution of the initial boundary value problem for the model (I) – classical elastic filtration model by Monte Carlo methods

3.1. Mathematical setting of Dirichlet problem

for this model has the following form:

$$\chi\Delta p(x,t) = \frac{\partial p(x,t)}{\partial t}, \tag{3.1}$$

$$p(x,t) = a(x), \text{while } \ t = 0, \tag{3.2}$$

$$p(x,t) = p_1(x,t) \ \text{ for } \ x \in \partial\Omega \times [0,T]. \tag{3.3}$$

After approximation only by time variable equation (3.1) has the form:

$$\Delta p^{n+1}(x) - a_1 p^{n+1}(x) = f^n(x), \tag{3.4}$$

where $a_1 = \dfrac{1}{\Delta\tau\chi}$, $f^n(x) = -\dfrac{1}{\Delta\tau\chi}p^n(x)$. Initial and boundary conditions (3.2) and (3.3):

$$p^0(x) = a(x), \quad x \in \Omega, \tag{3.5}$$

$$p^{n+1}(x) = p_1^{n+1}(x), \quad x \in \partial\Omega. \tag{3.6}$$

Problem (3.4) – (3.6) for a fixed time layer $n = 0, 1, \ldots, N-1$ can be considered as a Dirichlet problem for Helmholtz equation.

3.2. Mathematical setting of Neumann problem

for classical elastic filtration model:

$$\chi\Delta p(x,t) = \frac{\partial p(x,t)}{\partial t}, \tag{3.7}$$

$$p(x,t) = a(x), \text{while } \ t = 0, \tag{3.8}$$

$$\frac{\partial p(x,t)}{\partial \mathbf{n}} = p_2(x,t) \ \text{ for } \ x \in \partial\Omega \times [0,T], \tag{3.9}$$

where $\mathbf{n}$ is an internal normal. After approximation only by time variable t equation (3.7) has the form:

$$\Delta p^{n+1}(x) - a_1 p^{n+1}(x) = f^n(x), \tag{3.10}$$

where $a_1 = \dfrac{1}{\Delta\tau\chi}$, $f^n(x) = -\dfrac{1}{\Delta\tau\chi}p^n(x)$. Initial condition (3.8) has the same form as for approximated Dirichlet problem:

$$p^0(x) = a(x), \quad x \in \Omega, \tag{3.11}$$

and boundary Neumann condition (3.9):

$$\frac{\partial p^{n+1}(x)}{\partial \mathbf{n}} = p_2^{n+1}(x) \ \ x \in \partial\Omega. \tag{3.12}$$

Problem (3.10) – (3.12) for a fixed time layer $n = 0, 1, \ldots, N-1$ can be considered as a Neumann problem for Helmholtz equation.

3.3. Mathematical setting of mixed problem

for classical elastic filtration model:

$$\chi\Delta p(x,t) = \frac{\partial p(x,t)}{\partial t}, \tag{3.13}$$

$$p(x,t) = a(x), \text{while } \ t = 0, \tag{3.14}$$

$$\alpha_1 p(x,t) + \beta_1 \frac{\partial p(x,t)}{\partial \mathbf{n}} = p_3(x,t) \ \text{ for } \ x \in \partial\Omega \times [0,T], \tag{3.15}$$

where $\mathbf{n}$ is an internal normal. After approximation only by time variable t equation (3.13) has the form:

$$\Delta p^{n+1}(x) - a_1 p^{n+1}(x) = f^n(x), \tag{3.16}$$

where $a_1 = \dfrac{1}{\Delta\tau\chi}$, $f^n(x) = -\dfrac{1}{\Delta\tau\chi}p^n(x)$. Initial condition (3.14) has the same form as for approximated Dirichlet problem:

$$p^0(x) = a(x), \quad x \in \Omega, \tag{3.17}$$

and mixed boundary condition (3.15):

$$\alpha_1 p^{n+1}(x) + \beta_2 p^{n+1}(x) = p_3^{n+1}(x), \quad x \in \partial\Omega_\varepsilon, \tag{3.18}$$

where $\beta_2 = \beta_1\big(d(\partial\Omega) \cdot \nabla\big)$. Problem (3.16) – (3.18) for a fixed time layer $n = 0, 1, \ldots, N-1$ can be considered as a mixed problem for Helmholtz equation.

Problems described in 3.1, 3.2 and 3.3, that is Dirichlet, Neumann and mixed problems (3.4) – (3.6), (3.10) – (3.12) and (3.16) – (3.18) are solved by Monte Carlo methods algorithms in the same way as problems from 2, as for model **III** – Filtration in relaxationaly-compressed porous environment realized by the linear Darcy law.

4. Solution of the initial boundary value problem for the model (II) – the simplest model of filtration with a constant speed of disturbance spread by Monte Carlo methods

4.1. Mathematical setting of Dirichlet problem

for this model has the following form:

$$\chi \Delta p(x,t) = \frac{\partial p(x,t)}{\partial t} + \tau \frac{\partial^2 p(x,t)}{\partial t^2}, \tag{4.1}$$

$$p(x,t) = a(x), \text{while } \ t = 0, \tag{4.2}$$

$$\frac{\partial p(x,t)}{\partial t} = b(x), \text{while } \ t = 0, \tag{4.3}$$

$$p(x,t) = p_1(x,t) \ \text{ for } \ x \in \partial\Omega \times [0,T]. \tag{4.4}$$

After approximation only by time variable t equation (4.1) has the form:

$$\Delta p^{n+1}(x) - a_1 p^{n+1}(x) = f^n(x), \tag{4.5}$$

where $a_1 = \dfrac{\Delta\tau + \tau}{\Delta\tau^2\chi}$, $f^n(x) = -\dfrac{\Delta\tau + 2\tau}{\Delta\tau^2\chi} p^n(x) + \dfrac{\tau}{\Delta\tau^2\chi} p^{n-1}(x)$. Initial and boundary conditions (4.2), (4.3) and (4.4):

$$p^0(x) = a(x), \ \ x \in \Omega, \tag{4.6}$$

$$\frac{p^1(x) - p^0(x)}{\Delta\tau} = b(x), \quad x \in \Omega, \tag{4.7}$$

$$p^{n+1}(x) = p_1^{n+1}(x), \ \ x \in \partial\Omega. \tag{4.8}$$

Problem (4.5) – (4.8) for a fixed time layer $n = 0, 1, \ldots, N-1$ can be considered as a Dirichlet problem for Helmholtz equation.

4.2. Mathematical setting of Neumann problem

for this model has the following form:

$$\chi\Delta p(x,t) = \frac{\partial p(x,t)}{\partial t} + \tau\frac{\partial^2 p(x,t)}{\partial t^2}, \tag{4.9}$$

$$p(x,t) = a(x), \text{while } \ t = 0, \tag{4.10}$$

$$\frac{\partial p(x,t)}{\partial t} = b(x), \text{while } \ t = 0, \tag{4.11}$$

$$\frac{\partial p(x,t)}{\partial \mathbf{n}} = p_2(x,t) \ \text{ for } \ x \in \partial\Omega \times [0,T], \tag{4.12}$$

where $\mathbf{n}$ is an internal normal. After approximation only by time variable t equation (4.9) has the form:

$$\Delta p^{n+1}(x) - a_1 p^{n+1}(x) = f^n(x), \tag{4.13}$$

where $a_1 = \dfrac{\Delta\tau + \tau}{\Delta\tau^2\chi}$, $f^n(x) = -\dfrac{\Delta\tau + 2\tau}{\Delta\tau^2\chi}p^n(x) + \dfrac{\tau}{\Delta\tau^2\chi}p^{n-1}(x)$. Initial and boundary conditions (4.10), (4.11) and (4.12):

$$p^0(x) = a(x), \ \ x \in \Omega, \tag{4.14}$$

$$\frac{p^1(x) - p^0(x)}{\Delta\tau} = b(x), \quad x \in \Omega, \tag{4.15}$$

$$\frac{\partial p^{n+1}(x)}{\partial \mathbf{n}} = p_2^{n+1}(x), \quad x \in \partial\Omega. \tag{4.16}$$

Problem (4.13) – (4.16) for a fixed time layer $n = 0, 1, \ldots, N-1$ can be considered as a Neumann problem for Helmholtz equation.

4.3. Mathematical setting of mixed problem

for this model has the following form:

$$\chi\Delta p(x,t) = \frac{\partial p(x,t)}{\partial t} + \tau\frac{\partial^2 p(x,t)}{\partial t^2}, \tag{4.17}$$

$$p(x,t) = a(x), \text{while } \ t = 0, \tag{4.18}$$

$$\frac{\partial p(x,t)}{\partial t} = b(x), \text{while } \ t = 0, \tag{4.19}$$

$$\alpha_1 p(x,t) + \beta_1\frac{\partial p(x,t)}{\partial \mathbf{n}} = p_3(x,t) \ \text{ for } \ x \in \partial\Omega \times [0,T], \tag{4.20}$$

where $\mathbf{n}$ is an internal normal. After approximation only by time variable t equation (4.17) has the form:

$$\Delta p^{n+1}(x) - a_1 p^{n+1}(x) = f^n(x), \tag{4.21}$$

where $a_1 = \dfrac{\Delta\tau + \tau}{\Delta\tau^2\chi}$, $f^n(x) = -\dfrac{\Delta\tau + 2\tau}{\Delta\tau^2\chi}p^n(x) + \dfrac{\tau}{\Delta\tau^2\chi}p^{n-1}(x)$. Initial and boundary conditions (4.18), (4.19) and (4.20):

$$p^0(x) = a(x), \ \ x \in \Omega, \tag{4.22}$$

$$\frac{p^1(x) - p^0(x)}{\Delta\tau} = b(x), \quad x \in \Omega, \tag{4.23}$$

$$\alpha_1 p^{n+1}(x) + \beta_2 p^{n+1}(x) = p_3^{n+1}(x), \quad x \in \partial\Omega_\varepsilon, \tag{4.24}$$

where $\beta_2 = \beta_1\big(d(\partial\Omega)\cdot\nabla\big)$. Problem (4.21) – (4.24) for a fixed time layer $n = 0, 1, \ldots, N-1$ can be considered as a mixed problem for Helmholtz equation.

Problems described in 4.1, 4.2 and 4.3, that is Dirichlet, Neumann and mixed problems (4.5) – (4.8), (4.13) – (4.16) and (4.21) – (4.24) are solved by Monte Carlo methods algorithms in the same way as problems from 2, as for model **III** – Filtration in relaxationaly-compressed porous environment realized by the linear Darcy law.

5. Solution of the initial boundary value problem for the model (IV) – Model of filtration by the simplest unbalanced law in elastic porous environment by Monte Carlo methods

5.1. Mathematical setting of Dirichlet problem

for this model has the following form:

$$\chi\Delta\left(p(x,t) + \tau_p\frac{\partial p(x,t)}{\partial t}\right) = \frac{\partial}{\partial t}\left(p(x,t) + \tau_W\frac{\partial p(x,t)}{\partial t}\right), \tag{5.1}$$

$$p(x,t) = a(x), \text{while } \ t = 0, \tag{5.2}$$

$$\frac{\partial p(x,t)}{\partial t} = b(x), \text{while } \ t = 0, \tag{5.3}$$

$$p(x,t) = p_1(x,t) \ \text{ for } \ x \in \partial\Omega \times [0,T]. \tag{5.4}$$

After approximation only by time variable t equation (5.1) has the form:

$$\Delta p^{n+1}(x) - a_1 p^{n+1}(x) = f^n(x), \tag{5.5}$$

where $a_1 = \dfrac{\Delta\tau + \tau_W}{\Delta\tau\chi(\Delta\tau + \tau_p)}$, $f^n(x) = \dfrac{\tau_p}{\Delta\tau + \tau_p}\Delta p^n(x) - \dfrac{\Delta\tau + 2\tau_W}{\Delta\tau\chi(\Delta\tau + \tau_p)}p^n(x) + \dfrac{\tau_W}{\Delta\tau\chi(\Delta\tau + \tau_p)}p^{n-1}(x)$. Initial and boundary conditions (5.2), (5.3) and (5.4):

$$p^0(x) = a(x), \ \ x \in \Omega, \tag{5.6}$$

$$\frac{p^1(x) - p^0(x)}{\Delta\tau} = b(x), \quad x \in \Omega, \tag{5.7}$$

$$p^{n+1}(x) = p_1^{n+1}(x), \ \ x \in \partial\Omega. \tag{5.8}$$

Problem (5.5) – (5.8) for a fixed time layer $n = 0, 1, \ldots, N-1$ can be considered as a Dirichlet problem for Helmholtz equation.

5.2. Mathematical setting of Neumann problem
for this model has the following form:

$$\chi\Delta\left(p(x,t)+\tau_p\frac{\partial p(x,t)}{\partial t}\right)=\frac{\partial}{\partial t}\left(p(x,t)+\tau_W\frac{\partial p(x,t)}{\partial t}\right), \tag{5.9}$$

$$p(x,t)=a(x), \text{while } \ t=0, \tag{5.10}$$

$$\frac{\partial p(x,t)}{\partial t}=b(x), \text{while } \ t=0, \tag{5.11}$$

$$\frac{\partial p(x,t)}{\partial \mathbf{n}}=p_2(x,t) \ \text{ for } \ x\in\partial\Omega\times[0,T], \tag{5.12}$$

where $\mathbf{n}$ is an internal normal. After approximation only by time variable t equation (4.9) has the form:

$$\Delta p^{n+1}(x)-a_1p^{n+1}(x)=f^n(x), \tag{5.13}$$

where $a_1=\dfrac{\Delta\tau+\tau_W}{\Delta\tau\chi(\Delta\tau+\tau_p)}$, $f^n(x)=\dfrac{\tau_p}{\Delta\tau+\tau_p}\Delta p^n(x)-\dfrac{\Delta\tau+2\tau_W}{\Delta\tau\chi(\Delta\tau+\tau_p)}p^n(x)+\dfrac{\tau_W}{\Delta\tau\chi(\Delta\tau+\tau_p)}p^{n-1}(x)$. Initial and boundary conditions (5.10), (5.11) and (5.12):

$$p^0(x)=a(x), \ \ x\in\Omega, \tag{5.14}$$

$$\frac{p^1(x)-p^0(x)}{\Delta\tau}=b(x), \quad x\in\Omega, \tag{5.15}$$

$$\frac{\partial p^{n+1}(x)}{\partial \mathbf{n}}=p_2^{n+1}(x), \qquad x\in\partial\Omega. \tag{5.16}$$

Problem (5.13) – (5.16) for a fixed time layer $n=0,1,\ldots,N-1$ can be considered as a Neumann problem for Helmholtz equation.

5.3. Mathematical setting of mixed problem
for this model has the following form:

$$\chi\Delta\left(p(x,t)+\tau_p\frac{\partial p(x,t)}{\partial t}\right)=\frac{\partial}{\partial t}\left(p(x,t)+\tau_W\frac{\partial p(x,t)}{\partial t}\right), \tag{5.17}$$

$$p(x,t)=a(x), \text{while } \ t=0, \tag{5.18}$$

$$\frac{\partial p(x,t)}{\partial t}=b(x), \text{while } \ t=0. \tag{5.19}$$

$$\alpha_1p(x,t)+\beta_1\frac{\partial p(x,t)}{\partial \mathbf{n}}=p_3(x,t) \ \text{ for } \ x\in\partial\Omega\times[0,T], \tag{5.20}$$

where $\mathbf{n}$ is an internal normal. After approximation only by time variable t equation (5.17) has the form:

$$\Delta p^{n+1}(x)-a_1p^{n+1}(x)=f^n(x), \tag{5.21}$$

where $a_1=\dfrac{\Delta\tau+\tau_W}{\Delta\tau\chi(\Delta\tau+\tau_p)}$, $f^n(x)=\dfrac{\tau_p}{\Delta\tau+\tau_p}\Delta p^n(x)-\dfrac{\Delta\tau+2\tau_W}{\Delta\tau\chi(\Delta\tau+\tau_p)}p^n(x)+\dfrac{\tau_W}{\Delta\tau\chi(\Delta\tau+\tau_p)}p^{n-1}(x)$. Initial and boundary conditions (5.18), (5.19) and (5.20):

$$p^0(x)=a(x), \ \ x\in\Omega, \tag{5.22}$$

$$\frac{p^1(x)-p^0(x)}{\Delta\tau}=b(x),\quad x\in\Omega, \tag{5.23}$$

$$\alpha_1\,p^{n+1}(x)+\beta_2\,p^{n+1}(x)=p_3^{n+1}(x),\quad x\in\partial\Omega_\varepsilon, \tag{5.24}$$

where $\beta_2=\beta_1\big(d(\partial\Omega)\cdot\nabla\big)$. Problem (5.21) – (5.24) for a fixed time layer $n=0,1,\ldots,N-1$ can be considered as a mixed problem for Helmholtz equation.

Problems described in 5.1, 5.2 and 5.3, that is Dirichlet, Neumann and mixed problems (5.5) – (5.8), (5.13) – (5.16) and (5.21) – (5.24) are solved by Monte Carlo methods algorithms in the same way as problems from 2, as for model **III** – Filtration in relaxationaly-compressed porous environment realized by the linear Darcy law.

6. Solution of the initial boundary value problem for the model (III) – Filtration model in relaxationaly-compressed porous environment realized by the linear Darcy law, part 2 ($\beta_f=0$ and $\lambda_p=0$), by Monte Carlo methods

6.1. Mathematical setting of Dirichlet problem

for this model has the following form:

$$\chi\Delta\left(p(x,t)+\lambda_m\frac{\partial p(x,t)}{\partial t}\right)=\frac{\partial p(x,t)}{\partial t}, \tag{6.1}$$

$$p(x,t)=a(x),\text{while }\ t=0, \tag{6.2}$$

$$p(x,t)=p_1(x,t)\ \text{ for }\ x\in\partial\Omega\times[0,T]. \tag{6.3}$$

After approximation only by time variable t equation (6.1) has the form:

$$\Delta p^{n+1}(x)-a_1p^{n+1}(x)=f^n(x), \tag{6.4}$$

where $a_1=\dfrac{1}{\chi(\Delta\tau+\lambda_m)}$, $f^n(x)=\dfrac{\lambda_m}{\Delta\tau+\lambda_m}\Delta p^n(x)-\dfrac{1}{\chi(\Delta\tau+\lambda_m)}p^n(x)$. Initial and boundary conditions (6.2)and (6.3):

$$p^0(x)=a(x),\ \ x\in\Omega, \tag{6.5}$$

$$p^{n+1}(x)=p_1^{n+1}(x),\ \ x\in\partial\Omega. \tag{6.6}$$

Problem (6.4) – (6.6) for a fixed time layer $n=0,1,\ldots,N-1$ can be considered as a Dirichlet problem for Helmholtz equation.

6.2. Mathematical setting of Neumann problem

for this model has the following form:

$$\chi\Delta\left(p(x,t)+\lambda_m\frac{\partial p(x,t)}{\partial t}\right)=\frac{\partial p(x,t)}{\partial t}, \tag{6.7}$$

$$p(x,t)=a(x), \text{while } \ t=0, \tag{6.8}$$

$$\frac{\partial p(x,t)}{\partial \mathbf{n}}=p_2(x,t) \ \text{ for } \ x\in\partial\Omega\times[0,T], \tag{6.9}$$

where $\mathbf{n}$ is an internal normal. After approximation only by time variable t equation (6.7) has the form:

$$\Delta p^{n+1}(x)-a_1p^{n+1}(x)=f^n(x), \tag{6.10}$$

where $a_1=\dfrac{1}{\chi(\Delta\tau+\lambda_m)}$, $f^n(x)=\dfrac{\lambda_m}{\Delta\tau+\lambda_m}\Delta p^n(x)-\dfrac{1}{\chi(\Delta\tau+\lambda_m)}p^n(x)$. Initial and boundary conditions (6.8) and (6.9):

$$p^0(x)=a(x), \ \ x\in\Omega, \tag{6.11}$$

$$\frac{\partial p^{n+1}(x)}{\partial \mathbf{n}}=p_2^{n+1}(x), \qquad x\in\partial\Omega. \tag{6.12}$$

Problem (6.10) – (6.12) for a fixed time layer $n=0,1,\ldots,N-1$ can be considered as a Neumann problem for Helmholtz equation.

6.3. Mathematical setting of mixed problem

for this model has the following form:

$$\chi\Delta\left(p(x,t)+\lambda_m\frac{\partial p(x,t)}{\partial t}\right)=\frac{\partial p(x,t)}{\partial t}, \tag{6.13}$$

$$p(x,t)=a(x), \text{while } \ t=0, \tag{6.14}$$

$$\alpha_1p(x,t)+\beta_1\frac{\partial p(x,t)}{\partial \mathbf{n}}=p_3(x,t) \ \text{ for } \ x\in\partial\Omega\times[0,T], \tag{6.15}$$

where $\mathbf{n}$ is an internal normal. After approximation only by time variable t equation (6.13) has the form:

$$\Delta p^{n+1}(x)-a_1p^{n+1}(x)=f^n(x), \tag{6.16}$$

where $a_1=\dfrac{1}{\chi(\Delta\tau+\lambda_m)}$, $f^n(x)=\dfrac{\lambda_m}{\Delta\tau+\lambda_m}\Delta p^n(x)-\dfrac{1}{\chi(\Delta\tau+\lambda_m)}p^n(x)$. Initial and boundary conditions (6.14) and (6.15):

$$p^0(x)=a(x), \ \ x\in\Omega, \tag{6.17}$$

$$\alpha_1\,p^{n+1}(x)+\beta_2\,p^{n+1}(x)=p_3^{n+1}(x), \quad x\in\partial\Omega_\varepsilon, \tag{6.18}$$

where $\beta_2=\beta_1\big(d(\partial\Omega)\cdot\nabla\big)$. Problem (6.16) – (6.18) for a fixed time layer $n=0,1,\ldots,N-1$ can be considered as a mixed problem for Helmholtz equation.

Problems described in 6.1, 6.2 and 6.3, that is Dirichlet, Neumann and mixed problems ((6.4) – (6.6), (6.10) – (6.12) and (6.16) – (6.18) are solved by Monte Carlo methods algorithms in the same way as problems from 2, as for model **III** –

Filtration in relaxationaly-compressed porous environment realized by the linear Darcy law.

Remark 2. In all considered models after evaluating pressure $p(x,t)$ by Monte Carlo methods, the first derivatives of $p(x,t)$ ($\operatorname{grad}_x p(x,t)$) are also evaluated by Monte Carlo methods. Then we can evaluate a rate of filtration $\mathbf{W}(x,t)$ in all models by Monte Carlo methods. For example, for model **II**: vector equation

$$\tau\frac{\partial\mathbf{W}(x,t)}{\partial t}+\mathbf{W}(x,t)=-\frac{\kappa}{\mu}\operatorname{grad}_x p(x,t)$$

can be approximated only by t. Then we get:

$$\tau\frac{\mathbf{W}^{n+1}(x)-\mathbf{W}^n(x)}{\Delta\tau}+\mathbf{W}^n(x)=\mathbf{f}_1^n(x)$$

or

$$\mathbf{W}^{n+1}(x)=\Big(1-\frac{\Delta\tau}{\tau}\Big)\mathbf{W}^n(x)+\frac{\Delta\tau}{\tau}\mathbf{f}_1^n(x),\ \ n=0,1,\ldots,N-1,$$

where $\mathbf{W}^0(x)$ is known because of initial condition $\mathbf{W}(x,t)=\widetilde{\mathbf{W}}(x)$ for $t=0$, and function $\mathbf{f}_1^n(x)=-\frac{\kappa}{\mu}\operatorname{grad}_x p^n(x)$ is also evaluated by Monte Carlo methods function in point x.

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